IV Semester M.Sc. Examination, June 2016
(RNS)
MATHEMATICS
M-401 : Measure and Integration

Time : 3 Hours Max. Marks : 80

Instructions : *i) Answer* ***any five*** *questions choosing atleast* ***two*** *from each Part.*
ii) ***All*** *questions carry* ***equal*** *marks.*

PART – A

1. a) Define a Borel set, for any singleton set {x} prove that m*({x}) = 0. **4**

   b) Show that the Lebesgue measure of an interval is equal to its length. Hence prove that the interval [0,1] is uncountable. **8**

   c) Define a measurable set. Show that the union of two measurable sets are measurable. **4**

2. a) Show that $m_e(A) \geq m_i(A)$ for any set A. **3**

   b) State and prove countably additive property of Lebesgue measurable sets. **6**

   c) Let A be any subset of $\mathbb{R}$. If $E_1, E_2, \ldots E_n$ are disjoint Lebesgue measurable sets then prove that $m^*\left(A \cap \left(\bigcup_{i=1}^{n} E_i\right)\right) = \sum_{i=1}^{n} m^*(A \cap E_i)$ **7**

3. a) If f and g are two measurable real valued functions defined on the same domain then, prove that f + c, cf, f + g, f – g , f^2, fg are also measurable. **8**

   b) Let f be a measurable function and B be a Borel. Then prove that $f^{-1}(B)$ is a measurable set. **8**

4. a) Let E be a Lebesgue measurable set with finite measure. For a given $\in >0$, prove that there exists a finite union 'U' of open intervals such that $m(E \Delta U) < \in$ where $E \Delta U = (E - U) \cup (U - E)$. **7**

b) Let f be a measurable function and g be a function defined over a measurable set E. Such that f = g a.e. on E. Then prove that g is measurable. **4**

c) If a sequence $\{f_n\}$ converges in measure to f then prove the following :

i) $\{f_n\}$ converges in measure to every function g which is equivalent to f.

ii) The limit function f is unique a.e. **5**

PART – B

5. a) If f and g are bounded measurable function defined on a set is finite then

i) $\int_E af + bg = a\int_E f + b\int_E g$

ii) If f = g almost everywhere then $\int_E f = \int_E g$.

iii) If $f \leq g$ almost everywhere $\int_E f \leq \int_E g$ and $\left|\int_E f\right| \leq \int_E |f|$.

iv) If $A \leq f(x) \leq B$ then

$A.\, m(E) \int_E f \leq B.\, m(E)$.

v) If A and B are disjoint measurable sets then $\int_{A \cup B} f = \int_A f + \int_B f$. **10**

b) State and prove Fatou's Lemma. **6**

6. a) Let f be a non-negative function which is integrable over a set E. Then prove that for a given $\in > 0$ there is a $\partial > 0$ such that for every set $A \subset E$ with $mA < \delta$ are have $\int_A f < \varepsilon$. **6**

b) If f and g are integrable over E then prove that

i) The function cf is integrable over E and $\int_E cf = c\int_E f$.

ii) The function f + g is integrable over E and $\int_E f + g = \int_E f + \int_E g$. **4**

c) State and prove Lebesgue convergence theorem. **6**

7. a) Establish Vitali covering Lemma. **9**

b) Define a function of bounded variation. If f is a function of bounded variation on [a, b], then prove that $T_a^b = P_a^b + N_a^b$ and f (b) – f (a) = $P_a^b - N_a^b$. **7**

8. a) Prove that $L^p \left(1 \le p \le \infty\right)$ spaces are complete. **7**

b) State and prove Riesz representation theorem. **9**

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